

Simplex and MacDonald Codes over R_q

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Abstract

In this paper, we introduce the homogeneous weight and homogeneous Gray map over the ring $R_q = \mathbb{F}_2[u_1, u_2, \dots, u_q] / \langle u_i^2 = 0, u_i u_j = u_j u_i \rangle$ for $q \geq 1$. We also consider the construction of simplex and MacDonald codes of types α and β over this ring. Further, we study the properties of these codes such as their binary images and covering radius.

Key Words: Simplex codes, MacDonald codes, Gray map, Codes over rings, Lee weight, Homogeneous weight.

1 Introduction

Codes over rings have been of significant research interest since the pioneering work of Hammons et al. [10] on codes over \mathbb{Z}_4 . Many of their results have been extended to finite chain rings such as Galois rings and rings of the form $\mathbb{F}_2[u] / \langle u^m \rangle$. Recently, as a generalization of previous studies [14, 15], Dougherty et al. [5] considered codes over an infinite class of rings, denoted R_q . These rings are finite and commutative, but are not finite chain rings. Motivated by the importance of the simplex and MacDonald codes which have been defined over several finite commutative rings [1, 8, 9], in this work, we define the homogeneous weight over R_q and present simplex codes and MacDonald codes over this ring. The properties of these codes are studied, particularly the weight enumerators and covering radius. Further, the binary images of these codes are considered.

The remainder of this paper is organized as follows. In Section 2, some preliminary results are given concerning the ring R_q and codes over this ring. Further, we define the homogeneous weight and its Gray map. The simplex codes of type α and their properties and binary images are given in Section 3, while the simplex codes of type β and their properties and binary images are given in Section 4. In Section 5, the MacDonald codes of types α and β are presented along with their binary images. Section 6 presents the repetition codes and considers some properties of these codes, in particular the covering radius. Finally, in

Section 7 the covering radius of the Simplex and MacDonald codes of types α and β are studied.

2 Preliminaries

Let R be a finite commutative ring and R^n the set of all n -tuples over R . Hence, R^n is an R -module. A code C of length n over R is a non-empty subset of R^n . A submodule C of R^n is called a linear code, and a code C is called free if it is a free R -module. Let $|C|$ denote the cardinality of C . If $|C| = M$, then C is called an (n, M) code. For any two vectors (or codewords) $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in R^n$, the inner product is defined as

$$\langle x, y \rangle = \sum_{i=1}^n x_i y_i \in R.$$

Let $C \subseteq R^n$ be a code of length n over R . The dual code of C is defined as

$$C^\perp = \{x \mid \langle x, y \rangle = 0, \text{ for all } y \in C\}.$$

Let $q \geq 2$ be a positive integer. Then the ring $R_q = \mathbb{F}_2[u_1, u_2, \dots, u_q] / \langle u_i^2 = 0, u_i u_j = u_j u_i \rangle$ is given recursively by

$$R_q = \mathbb{F}_2[u_1, u_2, \dots, u_q] / \langle u_i^2 = 0, u_i u_j = u_j u_i \rangle = R_{q-1} + u_q R_{q-1}.$$

For every subset $A \subseteq \{1, 2, \dots, q\}$ we have

$$u_A = \prod_{i \in A} u_i,$$

with the convention that $u_\emptyset = 1$. Then all elements of R_q can be expressed by

$$\sum_{A \subseteq \{1, 2, \dots, q\}} c_A u_A, \text{ with } c_A \in \mathbb{F}_2.$$

The following lemmas proved by Dougherty et al. [5] gives some important properties of R_q .

Lemma 2.1 *The ring R_q is a local commutative ring with $|R_q| = 2^{2^q}$. The unique maximal ideal m_q consists of all non-units and $|m_q| = \frac{|R_q|}{2}$.*

Proposition 2.2

(i) *For any $a \in R_q$, we have*

$$a \cdot (u_1 u_2 \cdots u_q) = \begin{cases} 0 & \text{if } a \text{ is a non-unit,} \\ u_1 u_2 \cdots u_q & \text{if } a \text{ is a unit.} \end{cases}$$

(ii) For any unit $a \in R_q$ and $x \in R_q$, we have

$$a \cdot x = u_1 u_2 \cdots u_q \Leftrightarrow x = u_1 u_2 \cdots u_q.$$

We denote the set of units of R_q by $\mathfrak{U}(R_q)$ and non-units by $\mathfrak{D}(R_q)$. It is clear that

$$|\mathfrak{U}(R_q)| = |\mathfrak{D}(R_q)| = 2^{2^q-1} \text{ and } \mathfrak{U}(R_q) = \mathfrak{D}(R_q) + 1.$$

A linear code of length n over R_q is defined to be an R_q -submodule of R_q^n .

2.1 The Lee and Homogeneous Weights over R_q and the Gray Maps

2.1.1 The Lee Weight over R_q and the Gray Map

Let the order on the subsets of $\{1, 2, \dots, q\}$ be

$$\{1, 2, \dots, q\} = \{1, 2, \dots, q-1\} \cup \{q\}.$$

With this order, the Gray map is defined as follows

$$\Psi_{Lee} : R_q \rightarrow \mathbb{F}_2^{2^q},$$

with

$$\Psi_{Lee}(u_A) = (c_B)_{B \subset \{1, 2, \dots, q\}},$$

and

$$c_B = \begin{cases} 1 & \text{if } B \subset A, \\ 0 & \text{otherwise.} \end{cases}$$

We can extend Ψ_{Lee} to all elements of R_q and define the Lee weight of an element in R_q as the Hamming weight of its image. This is a linear distance preserving map from R_q^n to $\mathbb{F}_2^{2^q n}$. It follows immediately that

$$w_{Lee}(u_A) = 2^{|A|}.$$

Hence we have the following lemma.

Lemma 2.3 *If C is a linear code over R_q of length n , cardinality 2^k and minimum Lee weight d_{Lee} , then $\Psi_{Lee}(C)$ is a binary linear code with parameters $[2^{2^q} n, k, d_{Lee}]$.*

2.1.2 The Homogeneous Weight over R_q and the Gray Map

Several weights can be defined over rings. A weight on a code C over the ring R_q is called homogeneous if it satisfies the following assertions.

Definition 2.4 [6, p. 19] A real valued function w on the finite ring R_q is called a (left) homogeneous weight if $w(0) = 0$ and the following are true.

(i) For all $x, y \in R_q$, $R_q x = R_q y$ implies $w(x) = w(y)$.

(ii) There exists a real number η such that

$$\sum_{y \in R_x} w(y) = \eta |R_q x| \text{ for all } x \in R_q - \{0\}.$$

The number η is the average value of w on R_q , and from condition (i) we can deduce that η is constant on every non-zero principal ideal of R_q .

Honold [11] described the homogeneous weight on R_q in terms of generating characters.

Proposition 2.5 [11] Let R_q be a finite ring with generating character χ . Then every homogeneous weight on R_q is of the form

$$\begin{aligned} w &: R_q \rightarrow \mathbb{R} \\ x &\mapsto \gamma \left[1 - \frac{1}{|R_q^\times|} \sum_{u \in R_q^\times} \chi(xu) \right] \end{aligned}$$

.

The homogeneous weight on R_q will be obtained using Proposition 2.5. Recall from [5] that the following is a generating character for the ring R_q

$$\chi \left(\sum_{\substack{A \subseteq \{1, 2, \dots, q\} \\ c_\emptyset = 0 \vee A \neq \emptyset}} c_A u_A \right) = (-1)^{wt(c)},$$

where $wt(c)$, denotes the Hamming weight of the \mathbb{F}_2 -coordinate vector of the element in the basis $\{u_A; A \subseteq \{1, 2, \dots, q\}\}$. We then have

$$\begin{aligned} \chi(0) &= 1 \\ \chi(1) &= \chi(u_1) = \dots = \chi(u_q) = \chi(u_1 u_2) = \dots = \chi(u_1 u_2 \dots u_q) = -1 \\ \chi(1 + u_1) &= \chi(u_1 + u_2) = \dots = \chi(u_q + u_1 u_2 \dots u_q) = 1 \\ \chi(1 + u_1 + u_2) &= \chi(u_1 + u_2 + u_3) = \dots = \chi(u_{q-1} + u_q + u_1 u_2 \dots u_q) = -1 \\ &\vdots \\ \chi \left(1 + \sum_{\substack{A \subseteq \{1, 2, \dots, q\} \\ c_\emptyset = 0 \vee A \neq \emptyset}} c_A u_A \right) &= 1. \end{aligned}$$

The following Lemma from [11, Theorem 2] will be key in proving the main theorem concerning the homogeneous weight on R_q .

Lemma 2.6 *Let x be an element in R_q such that $x \neq 0$ and $x \neq u_1 u_2 \cdots u_q$. Then*

$$\sum_{a \in R_q} \chi(a \cdot x) = 0.$$

Theorem 2.7 *The homogeneous weight on R_q is*

$$w_{hom}(x) = \begin{cases} 0 & \text{if } x = 0, \\ 2\gamma & \text{if } x = u_1 u_2 \cdots u_q, \\ \gamma & \text{otherwise.} \end{cases}$$

Proof. Let $x = u_1 u_2 \cdots u_q$. Then by Proposition 2.2, $a \cdot x = x$ for all $a \in \mathfrak{U}(R_q)$, so $\chi(a \cdot x) = -1$ for all $a \in \mathfrak{U}(R_q)$. Hence, by Proposition 2.5 we have

$$w_{hom}(x) = \gamma \left[1 - \frac{1}{|\mathfrak{U}(R_q)|} \sum_{a \in \mathfrak{U}(R_q)} (-1) \right] = 2\gamma.$$

If $x \neq 0$ and $x \neq u_1 u_2 \cdots u_q$, then by Lemma 2.6 we have $\sum_{a \in R_q} \chi(a \cdot x) = 0$. Thus we obtain

$$w_{hom}(x) = \gamma \left[1 - \frac{1}{|\mathfrak{U}(R_q)|} 0 \right] = \gamma.$$

□

The homogeneous weight for a codeword $x = (x_1, x_2, \dots, x_n) \in R_q^n$ is defined as

$$w_{hom}(x_i) = \begin{cases} 0 & \text{if } x_i = 0, \\ 2^{q+1} & \text{if } x_i = u_1 u_2 \cdots u_q, \\ 2^q & \text{otherwise.} \end{cases}$$

The corresponding Gray map is given by

$$\Psi_{hom} : R_q \rightarrow \mathbb{F}_2^{2^{q+1}}$$

where

$$\begin{aligned} \Psi_{hom}(0) &= 0000 \cdots 00 \\ \Psi_{hom}(1) &= 0101 \cdots 01 \\ \vdots &\quad \quad \quad \vdots \quad \quad \quad \vdots \\ \Psi_{hom}\left(\sum_{\substack{A \subseteq \{1, 2, \dots, q\} \\ c_\emptyset = 0 \vee A \neq \emptyset}} c_A u_A\right) &= 1111 \cdots 11 \end{aligned}$$

Hence the following lemma holds.

Lemma 2.8 *If C is a linear code over R_q of length n , cardinality 2^k and minimum homogeneous weight d_{hom} , then $\Psi_{hom}(C)$ is a binary linear code with parameters $[2^{2^{q+1}}n, k, d_{hom}]$.*

The following definition gives the Hamming, Lee and homogeneous weight distributions.

Definition 2.9 [9] For every $1 \leq i \leq n$, let $A_{Ham}(i)$, $A_{Lee}(i)$ and $A_{hom}(i)$ be the number of codewords of Hamming, Lee and homogeneous weight i in C , respectively. Then

$$\begin{aligned} & (A_{Ham}(0), A_{Ham}(1), \dots, A_{Ham}(n)), \\ & (A_{Lee}(0), A_{Lee}(1), \dots, A_{Lee}(n)), \\ & \text{and} \\ & (A_{hom}(0), A_{hom}(1), \dots, A_{hom}(n)), \end{aligned}$$

are called the Hamming, Lee and homogeneous weight distributions of C , respectively.

In [4], the *torsion* code of a code C over R_q was defined as

$$Tor_A(C) = \{v \in \mathbb{F}_2^n; u_A v \in C, A \subset \{1, \dots, 2^q\}\}. \quad (1)$$

$Tor_\emptyset(C) = \{v \in \mathbb{F}_2^n; u_\emptyset v \in C, A = \emptyset\}$ is called the residue code and is often denoted by $Res(C) = \{u \in \mathbb{F}_2^n; \exists v \in \mathbb{F}_2^n; u + u_A v \in C\}$. In general, we have the following tower of codes

$$Tor_\emptyset(C) \subseteq Tor_{\{i\} \subset \{1, \dots, 2^q\}}(C) \subseteq \dots \subseteq Tor_{\{1, \dots, 2^q\}}(C). \quad (2)$$

Hence for a code C over R_q

$$|C| = |Tor_\emptyset(C)| |Tor_{\{i\} \subset \{1, \dots, 2^q\}}(C)| \dots |Tor_{\{1, \dots, 2^q\}}(C)|.$$

Before presenting the simplex codes of types α and β , we define the 2-dimension of a code C . In [13], the authors presented the p -dimension for finitely generated modules over \mathbb{Z}_{p^s} . Using this result, we define the 2-dimension of a code C over R_q as follows. A subset S of C is a 2-basis for the linear code C over R_q if S is 2-linearly independent and C is the 2-span of S . The number of vectors in a 2-basis for C is called the 2-dimension of C .

2.2 The Covering Radius

The covering radius of a code is defined as the smallest integer r such that all vectors in the space are within distance r of some codeword. The covering radius of a code C over R_q is then

$$r_{Lee}(C) = \max_{v \in R_q^n} \{d(v, C)\} \text{ and } r_{hom}(C) = \max_{v \in R_q^n} \{d(v, C)\},$$

for the Lee and homogeneous weights, respectively. It is easy to see that $r_{Lee}(C)$ and $r_{hom}(C)$ are the minimum values of r_{Lee} and r_{hom} such that

$$R_q^n = \cup_{c \in C} S_{r_{Lee}}(c) \text{ and } R_q^n = \cup_{c \in C} S_{r_{hom}}(c),$$

respectively, where

$$S_{r_{Lee}}(u) = \{v \in R_q^n; d(u, v) \leq r_{Lee}\} \text{ and } S_{r_{hom}}(u) = \{v \in R_q^n; d(u, v) \leq r_{hom}\}.$$

Proposition 2.10 [2, Proposition 3.2] Let C be a code over R_q^n and $\Psi_{Lee}(C)$ the Gray map image of C . Then $r_{Lee}(C) = r_{Ham}(\Psi_{Lee}(C))$.

Proposition 2.11 If C_0 and C_1 are codes over R_q^n generated by matrices G_0 and G_1 , respectively, and if C is the code generated by

$$G = \left[\begin{array}{c|c} 0 & G_1 \\ \hline G_0 & A \end{array} \right],$$

then $r_d(C) \leq r_{d_0}(C_0) + r_{d_1}(C_1)$ and the covering radius of C_c (the concatenation of C_0 and C_1) satisfies the following inequality $r_d(C_c) \geq r_{d_0}(C_0) + r_{d_1}(C_1)$ for all distances d over R_q^n .

Proof. See [3, Part D]. □

3 Simplex Codes of Type α

Let q and k be positive integers with $q \geq 1$, and let $G_{(q,k)}^\alpha$ be the matrix of size $k \times 2^{2^q \cdot k}$ defined inductively by

$$G_{(q,k)}^\alpha = \left(\begin{array}{c|c|c} G_{(q,k-1)}^\alpha & \cdots & G_{(q,k-1)}^\alpha \\ \hline 0_{2^{2^q \cdot (k-1)}} & \cdots & \left(1 + \sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset = 0 \vee A \neq \emptyset}} c_A u_A \right) \times 1_{2^{2^q \cdot (k-1)}} \end{array} \right), \quad (3)$$

for $k \geq 2$, where

$$G_{(q,1)}^\alpha = \left(0 \ 1 \ u_1 \ \cdots \ \left(1 + \sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset = 0 \vee A \neq \emptyset}} c_A u_A \right) \right),$$

is a matrix with one row and 2^{2^q} columns containing all the elements of R_q . The columns of $G_{(q,k)}^\alpha$ consist of all distinct k -tuples over R_q . The code $S_{(q,k)}^\alpha$ generated by $G_{(q,k)}^\alpha$ is called the simplex code of type α over R_q . This code has length $2^{2^q k}$ and 2-dimension $2^q k$.

Remark 3.1 If A_{k-1} denotes the $2^{2^q \cdot (k-1)} \times 2^{2^q \cdot (k-1)}$ matrix consisting of all codewords in $S_{(q,k-1)}^\alpha$, and J is the matrix with all elements equal to 1, then $S_{(q,k)}^\alpha$ is generated by the

$2^{2^q \cdot k} \times 2^{2^q \cdot k}$ matrix

$$\begin{bmatrix} A_{k-1} & A_{k-1} & \cdots & A_{k-1} \\ A_{k-1} & J + A_{k-1} & \cdots & \left(1 + \sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset = 0 \vee A \neq \emptyset}} c_A u_A\right) J + A_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k-1} & \left(1 + \sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset = 0 \vee A \neq \emptyset}} c_A u_A\right) J + A_{k-1} & \cdots & J + A_{k-1} \end{bmatrix}. \quad (4)$$

Remark 3.2 If l_1, l_2, \dots, l_k are the rows of $G_{(q,k)}^\alpha$, then

1. $w_{Ham}(l_i) = 3 \cdot 5 \cdot 17 \cdot 257 \cdots 2^{2^q \cdot k - 2^q}$, $w_{Ham}(u_1 l_i) = w_{Ham}(u_2 l_i) = \dots = w_{Ham}(u_q l_i) = 3 \cdot 5 \cdot 17 \cdot 257 \cdots 2^{2^q \cdot k - 2^{q-1}}$, $w_{Ham}(u_1 u_2 \dots u_q l_i) = 2^{2^q \cdot k - 1}$.
2. $w_{Lee}(l_i) = w_{Lee}(u_1 l_i) = w_{Lee}(u_2 l_i) = \dots = w_{Lee}(u_1 u_2 \dots u_q l_i) = 2^{2^q \cdot k + (q-1)}$.
3. $w_{hom}(l_i) = w_{hom}(u_1 l_i) = w_{hom}(u_2 l_i) = \dots = w_{hom}(u_1 u_2 \dots u_q l_i) = 2^{2^q k}$.

In the matrix $G_{(q,k)}^\alpha$, it is clear that each element of R_q appears $2^{2^q \cdot (k-1)}$ times in every row. Thus we have the following lemma.

Lemma 3.3 Let $c \in S_{(q,k)}^\alpha$ be nonzero. If one coordinate of c is a unit then every element of R_q occurs $2^{2^q \cdot (k-1)}$ times as a coordinate of c .

Proof. From Remark 3.1, any $x \in S_{(q,k-1)}^\alpha$ gives the following codewords of $S_{(q,k)}^\alpha$

$$\begin{aligned} c_1 &= (x|x|x|\cdots|x) \\ c_2 &= \left(x|1+x|u_1+x|\cdots|\left(1 + \sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset = 0 \vee A \neq \emptyset}} c_A u_A\right) + x \right) \\ &\vdots \\ c_{2^{2^q}} &= \left(x|\left(1 + \sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset = 0 \vee A \neq \emptyset}} c_A u_A\right) + x|\cdots|x \right). \end{aligned}$$

The result then follows by induction on k and Remark 3.1. □

To obtain the torsion codes over R_q , it is necessary to introduce the binary simplex codes of type α and β .

The binary simplex code of type α , denoted by S_k , has parameters $[2^k; k; d_{Ham} = 2^{k-1}]$ and generator matrix

$$G_k = \left(\begin{array}{c|c} 00 \cdots 0 & 11 \cdots 1 \\ \hline G_{k-1} & G_{k-1} \end{array} \right), \quad (5)$$

for $k \geq 2$, where $G_1 = (0|1)$.

The binary simplex code of type β , denoted by \hat{S}_k , has parameters $[2^k - 1; k; d_{Ham} = 2^{k-1}]$ and generator matrix

$$\hat{G}_k = \left(\begin{array}{c|c} 11 \cdots 1 & 00 \cdots 0 \\ \hline G_{k-1} & \hat{G}_{k-1} \end{array} \right), \quad (6)$$

for $k \geq 3$, where

$$\hat{G}_2 = \left(\begin{array}{c|c} 11 & 0 \\ \hline 01 & 1 \end{array} \right).$$

Lemma 3.4 *The torsion code of $S_{(q,k)}^\alpha$ is the concatenation of $2^{(2^q-1)k}$ S_k codes.*

Proof. The torsion code of $S_{(q,k)}^\alpha$ is the set of codewords obtained by replacing $u_1 u_2 \cdots u_q$ with 1 in all $u_1 u_2 \cdots u_q$ -linear combinations of the rows of $u_1 \cdots u_q G_{(q,k)}^\alpha$ (where $G_{(q,k)}^\alpha$ is the generator matrix of $S_{(q,k)}^\alpha$ defined in (3)). The proof is by induction on k . For $k = 2$, the result is true. If $u_1 u_2 \cdots u_q G_{(q,k-1)}^\alpha$ is the matrix obtained by the concatenation of $2^{(2^q-1)(k-1)}$ copies of the matrix $u_1 u_2 \cdots u_q G_{k-1}$, then $u_1 u_2 \cdots u_q G_{(q,k)}^\alpha$ takes the form

$$\left(\begin{array}{c|c|c} u_1 u_2 \cdots u_q G_{k-1} \cdots u_1 u_2 \cdots u_q G_{k-1} & \cdots & u_1 u_2 \cdots u_q G_{k-1} \cdots u_1 u_2 \cdots u_q G_{k-1} \\ \hline 0_{2^{2^q \cdot (k-1)}} & \cdots & (u_1 u_2 \cdots u_q) \times 1_{2^{2^q \cdot (k-1)}} \end{array} \right). \quad (7)$$

Grouping the columns based on (5), we obtain the result. \square

For $q \geq 2$, we define the following linear homomorphism

$$\Gamma_q : R_q \rightarrow R_{q-1} \\ 1 + \sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset = 0 \vee A \neq \emptyset}} c_A u_A \mapsto \Gamma_q \left(1 + \sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset = 0 \vee A \neq \emptyset}} c_A u_A \right),$$

where

$$\Gamma_q \left(1 + \sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset = 0 \vee A \neq \emptyset}} c_A u_A \right) = 1 + \sum_{\substack{A \subseteq \{1,2,\dots,q-1\} \\ c_\emptyset = 0 \vee A \neq \emptyset}} c_A u_A.$$

We have

$$Im(\Gamma_q) = R_{q-1},$$

and for n a positive integer this homomorphism can be extended to R_q^n

$$\Gamma_q : R_q^n \longrightarrow R_{q-1}^n.$$

Theorem 3.5 *Let $S_{(q,k)}^\alpha$ be the simplex code of type α over R_q . Then $\Gamma_q(S_{(q,k)}^\alpha)$ is the concatenation of $2^{2^{q-1}k}$ simplex codes of type α over R_{q-1} .*

Proof. If $G_{(q,k)}^\alpha$ is a generator matrix of the simplex code $S_{(q,k)}^\alpha$ of type α over R_q , then $\Gamma_q(G_{(q,k)}^\alpha)$ has the form

$$\Gamma_q(G_{(q,k)}^\alpha) = \left(\overbrace{G_{(q-1,k)}^\alpha \mid G_{(q-1,k)}^\alpha \mid \cdots \mid G_{(q-1,k)}^\alpha}^{2^{2^{q-1}k}} \right),$$

where

$$G_{(q-1,k)}^\alpha = \left(\begin{array}{c|c|c|c} G_{(q-1,k-1)}^\alpha & G_{(q-1,k-1)}^\alpha & \cdots & G_{(q-1,k-1)}^\alpha \\ \hline 0_{2^{2^{q-1}(k-1)}} & 1_{2^{2^{q-1}(k-1)}} & \cdots & \left(1 + \sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset = 0 \vee A \neq \emptyset}} c_A u_A \right) \times 1_{2^{2^{q-1}(k-1)}} \end{array} \right),$$

is a generator matrix of the simplex code of type α over R_{q-1} . □

Theorem 3.6 *If $S_{(q,k)}^\alpha$ is a simplex code of type α over R_q , then*

$$\Gamma_q \left(\Gamma_{q-1} \cdots \left(\Gamma_2 \left(S_{(2,k)}^\alpha \right) \right) \right) = \left(\overbrace{S_{(1,k)}^\alpha S_{(1,k)}^\alpha \cdots S_{(1,k)}^\alpha}^{2^{2^q \left(\frac{q-1}{q} \right) k}} \right),$$

is the concatenation of $2^{2^q \left(\frac{q-1}{q} \right) k}$ $S_{(1,k)}^\alpha$ codes where $S_{(1,k)}^\alpha$ is the simplex code of type α over R_1 .

Proof. The proof is by induction on q and Theorem 3.6. For $q = 2$, if $G_{(2,k)}^\alpha$ is a generator matrix of the simplex code over R_2 , then

$$\Gamma_2(G_{(2,k)}^\alpha) = \left(\overbrace{G_{(1,k)}^\alpha \mid G_{(1,k)}^\alpha \mid \cdots \mid G_{(1,k)}^\alpha}^{2^{2k}} \right),$$

where $G_{(1,k)}^\alpha$ is a generator matrix of the simplex code over R_1 . If

$$\Gamma_{q-1} \left(\Gamma_{q-2} \cdots \left(\Gamma_2 \left(G_{(2,k)}^\alpha \right) \right) \right) = \left(2^{2^{q-1}k} \cdots \left(2^{2^2k} G_{(1,k)}^\alpha \right) \right),$$

is the generator matrix obtained by the concatenation of $2^{2^{(q-2)}(\frac{q+1}{2})k}$ simplex codes of type α over R_1 . Then

$$\Gamma_q (\Gamma_{q-1} \cdots (\Gamma_2 (G_{(2,k)}^\alpha))) = \left(2^{2^{qk}} \cdots (2^{2^{2k}} G_{(1,k)}^\alpha) \right) = \left(\overbrace{G_{(1,k)}^\alpha \mid G_{(1,k)}^\alpha \mid \cdots \mid G_{(1,k)}^\alpha}^{2^{2^q(\frac{q-1}{2})k}} \right).$$

□

Let $S_0 = \{0\}$, $S_1 = \{0, u_1 u_2 \cdots u_q\}, \dots$, $S_{q-1} = \{0, u_1, u_2, \dots, u_1 u_2 \cdots u_q\}$, and $S_q = R_q$. Note that S_{q-1} is the set of all zero divisors of R_q . A codeword $c = (c_1, c_2, \dots, c_n) \in S_{(q,k)}^\alpha$ is said to be of *type* m , $0 \leq m \leq q$, if all of its components belong to the set S_m . From $G_{(q,k)}^\alpha$, we have that each element of R_q occurs equally often in every row of $G_{(q,k)}^\alpha$.

To determine the Hamming, Lee and homogeneous weight distributions of $S_{(q,k)}^\alpha$, the number of codewords of type m in $S_{(q,k)}^\alpha$, $0 \leq m \leq q$, must be determined. For this, we define the matrix D_i as

$$D_0 = \begin{pmatrix} u_1 u_2 \cdots u_q l_1 \\ u_1 u_2 \cdots u_q l_2 \\ \vdots \\ u_1 u_2 \cdots u_q l_k \end{pmatrix}, D_1 = \begin{pmatrix} u_1 \cdots u_{q-1} l_1 \\ u_1 \cdots u_q l_1 \\ u_1 \cdots u_{q-1} l_2 \\ u_1 \cdots u_q l_2 \\ \vdots \\ u_1 \cdots u_{q-1} l_k \\ u_1 \cdots u_q l_k \end{pmatrix}, \dots, D_q = \begin{pmatrix} l_1 \\ u_1 l_1 \\ \vdots \\ u_1 \cdots u_q R_1 \\ l_2 \\ u_1 l_2 \\ \vdots \\ u_1 \cdots u_q l_2 \\ \vdots \\ l_k \\ u_1 l_k \\ \vdots \\ u_1 \cdots u_q l_k \end{pmatrix},$$

where l_i is the i^{th} row of $G_{(q,k)}^\alpha$. Let $C^{(m)}$ be the subcode of C generated by the rows of D_m . We then have that

$$C^{(0)} \subset C^{(2)} \subset \cdots \subset C^{(q)}.$$

Note that $C^{(m)}$ has 2^{mk} codewords and the matrix D_q generates $S_{(q,k)}^\alpha$. For $0 \leq m \leq q$, the codewords of type m occur $2^{mk} - 2^{(m-1)k}$ times in $S_{(q,k)}^\alpha$. This proves the following lemma.

Lemma 3.7 *For $0 \leq m \leq q$, the number of codewords of type m in $S_{(q,k)}^\alpha$ is $2^{(m-1)k}(2^k - 1)$.*

Theorem 3.8 *The Hamming, Lee and homogeneous weight distributions of $S_{(q,k)}^\alpha$ are*

- (i) $A_{Ham}(0) = 1, A_{Ham}(2^{(2^q k - m)}(2^m - 1)) = 2^{(m-1)k}(2^m - 1), \text{ for } 0 \leq m \leq q.$
- (ii) $A_{Lee}(0) = 1, A_{Lee}(2^{2^q k + (q-1)}) = 2^{2^q k} - 1.$
- (iii) $A_{hom}(0) = 1, A_{hom}(2^{2^q k}) = 2^{2^q k} - 1.$

Proof. Let $c \in S_{(q,k)}^\alpha$ be a codeword of type $m \neq 0$. Then by Lemma 3.7

$$A_{Ham}(2^{2^q - m}(2^m - 1)) = 2^{(m-1)k}(2^m - 1),$$

for $m = 0$, and $A_{Ham}(0) = 1$. Further, by Lemma 3.3, $A_{Lee}(c) = 2^{2^q k} - 1$ which is independent of m , so all codewords of type $m \neq 0$ have the same Lee and homogeneous weights. \square

3.1 Binary Gray Images of Simplex Codes of Type α

The binary images of the simplex code $S_{(q,k)}^\alpha$ over R_q are given in the following two theorems.

Theorem 3.9 *Let $S_{(q,k)}^\alpha$ be the simplex code over R_q of length $2^{2^q k}$, 2-dimension $2^q k$, and minimum Lee weight d_{Lee} . Then $\Psi_{Lee}(S_{(q,k)}^\alpha)$ is the concatenation of $2^{(2^q - 1)k + q}$ binary simplex codes with parameters $[2^{2^q k + q}; k; d_{Ham} = 2^{2^q k + q - 1}]$.*

Proof. Let $G_{(q,k)}^\alpha$ be a generator matrix of the simplex code $S_{(q,k)}^\alpha$ over R_q . Then $\Psi_{Lee}(G_{(q,k)}^\alpha)$ has the form

$$\Psi_{Lee}(G_{(q,k)}^\alpha) = \left(\overbrace{G_k \mid G_k \mid \cdots \mid G_k}^{2^{(2^q - 1)k + q}} \right),$$

where G_k is a generator matrix of the binary simplex code S_k . The result then follows by induction on k . \square

Theorem 3.10 *Let $S_{(q,k)}^\alpha$ be the simplex code over R_q of length $2^{2^q k}$, 2-dimension $2^q k$, and minimum homogeneous weight d_{hom} . Then $\Psi_{hom}(S_{(q,k)}^\alpha)$ is the concatenation of $2^{(2^q - 1)k + q + 1}$ binary simplex codes with parameters $[2^{2^q k + q + 1}; k; d_{Ham} = 2^{2^q k + q}]$.*

Proof. The proof is similar to that of Theorem 3.9. \square

4 Simplex Codes of Type β

Let $G_{(q,k)}^\beta$ be the matrix of size $k \times 2^{(2^q-1)(k-1)}(2^k-1)$ defined by

$$G_{(q,k)}^\beta = \left(\begin{array}{c|c|c|c} 1_{2^{2^q \cdot (k-1)}} & 0_{2^{2^q(k-2)}(2^{k-1}-1)} & \cdots & \left(\sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset=0 \vee A \neq \emptyset}} c_A u_A \right)_{2^{2^q(k-2)}(2^{k-1}-1)} \\ \hline G_{(q,k-1)}^\alpha & G_{(q,k-1)}^\beta & \cdots & G_{(q,k-1)}^\beta \end{array} \right),$$

for $k > 2$ and

$$G_{(q,2)}^\beta = \left(\begin{array}{c|c|c|c} 1_{2^{2^q \cdot (k-1)}} & 0 & \cdots & \sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset=0 \vee A \neq \emptyset}} c_A u_A \\ \hline 0 \ 1 \ \cdots \left(1 + \sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset=0 \vee A \neq \emptyset}} c_A u_A \right) & 1 & \cdots & 1 \end{array} \right),$$

where $G_{(q,k-1)}^\alpha$ is a generator matrix of $S_{(q,k-1)}^\alpha$.

Remark 4.1 Let A_{k-1} (B_{k-1}) denote the array of codewords in $S_{(q,k-1)}^\alpha$ ($S_{(q,k-1)}^\beta$), and J the matrix of all 1's. Then the array of codewords of $S_{(q,k)}^\beta$ is given by the following matrix

$$\left[\begin{array}{ccc} A_{k-1} & B_{k-1} & \cdots & B_{k-1} \\ J + A_{k-1} & B_{k-1} & \cdots & \left(\sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset=0 \vee A \neq \emptyset}} c_A u_A \right) J + B_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \left(1 + \sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset=0 \vee A \neq \emptyset}} c_A u_A \right) J + A_{k-1} & B_{k-1} & \cdots & \left(\sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset=0 \vee A \neq \emptyset}} c_A u_A \right) J + B_{k-1} \end{array} \right].$$

Let $\mathfrak{U}(\mathfrak{R}_q)$ and $\mathfrak{D}(\mathfrak{R}_q)$ denote the set of units and the set of zero divisors of R_q , respectively. The following proposition provides the weight distributions of $S_{(q,k)}^\beta$.

Proposition 4.2 For $1 \leq j \leq k$, let l_j be the j th row of $G_{(q,k)}^\beta$. Then we have

(i) $\sum_{i \in \mathfrak{U}(\mathfrak{R}_q)} w_i = 2^{2^q \cdot (k-1)}$ and each zero divisor in R_q appears $2^{(2^q-1)(k-2)}(2^{k-1}-1)$ times in l_j .

(ii) $w_{Ham}(l_j) = 2^{(2^q-1)(k-1)-2^q} (3 \cdot 5 \cdot 17 \cdot 257 \cdots (2^k-1) + 1)$.

1. $w_{Lee}(l_1) = 2^{2^q(k-1)} + 2^{2^q \cdot k - (2^q-1)} - 2^{4k-(2^q-2)}$.

(iii) $w_{hom}(l_j) = 2^{(2^q-1)k-1} (2^k-1)$.

Proof. The proof follows from the definition of l_j . □

The following proposition gives the structure of the codewords of $S_{(q,k)}^\beta$.

Proposition 4.3 *Consider a codeword $c \in S_{(q,k)}^\beta$. If one coordinate of c is a unit then $\sum_{i \in \mathcal{U}(\mathfrak{R}_q)} w_i = 2^{2^q \cdot (k-1)}$, and each zero divisor in R_q appears $2^{(2^q-1) \cdot (k-2)}(2^{k-1} - 1)$ times in c .*

Proof. By Remark 4.1, there exists $x_1 \in S_{(q,k-1)}^\alpha$ and $x_2 \in S_{(q,k-1)}^\beta$ such that c takes one of following 2^{2^q} forms

$$\begin{aligned} c_1 &= (x_1 | x_2 | x_2 | \cdots | x_2) \\ c_2 &= \left(1 + x_1 | x_2 | u_1 + x_2 | \cdots | \left(\sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset = 0 \vee A \neq \emptyset}} c_A u_A \right) + x_2 \right) \\ &\vdots \\ c_{2^{2^q}} &= \left(\left(1 + \sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset = 0 \vee A \neq \emptyset}} c_A u_A \right) + x_1 | \cdots | \left(\sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset = 0 \vee A \neq \emptyset}} c_A u_A \right) + x_2 \right). \end{aligned}$$

The result then follows by induction on k . □

Lemma 4.4 *The torsion code of $S_{(q,k)}^\beta$ is the concatenation of $2^{(2^q-1) \cdot (k-2)}$ binary simplex codes of type β denoted by \widehat{S}_k .*

Proof. The proof is similar to that of Lemma 3.4. □

Theorem 4.5 *The Hamming and homogeneous weight distribution of $S_{(q,k)}^\beta$ are*

$$(i) \ A_{Ham}(0) = 1, \ A_{Ham}(2^{(2^q-1)(k-1)}[(2^{k-m}(2^m - 1) + (2^{1-m} - 1)]) = 2^{(m-1)k}(2^m - 1), \\ 0 \leq m \leq q.$$

$$(ii) \ A_{hom}(0) = 1, \ A_{hom}(2^{(2^q-1)k-1}(2^k - 1)) = 2^k(2^{(2^q-1)k} - 1).$$

Proof. The proof is similar to that of Theorem 3.8. □

Theorem 4.6 *Let $S_{(q,k)}^\beta$ be the simplex code of type β over R_q . Then $\Gamma_q(S_{(q,k)}^\beta)$ is the concatenation of $2^{2^{q-1}k}$ simplex codes of type β over R_{q-1} .*

Proof. If $G_{(q,k)}^\beta$ is a generator matrix of the simplex code of type β over R_q , then $\Gamma_q(G_{(q,k)}^\beta)$ has the form

$$\Gamma_q(G_{(q,k)}^\beta) = \left(\overbrace{G_{(q-1,k)}^\beta \mid G_{(q-1,k)}^\beta \mid \cdots \mid G_{(q-1,k)}^\beta}^{2^{2k}} \right),$$

where $G_{(q-1,k)}^\beta$ is a generator matrix of the simplex code $S_{(q,k-1)}^\beta$ of type β over R_{q-1} . \square

Theorem 4.7 *If $S_{(q,k)}^\beta$ is the simplex code of type β over R_q , then*

$$\Gamma_q \left(\Gamma_{q-1} \cdots \left(\Gamma_2 \left(S_{(2,k)}^\beta \right) \right) \right) = \left(\overbrace{S_{(1,k)}^\beta S_{(1,k)}^\beta \cdots S_{(1,k)}^\beta}^{2^{2^q \left(\frac{q-1}{2} \right)_k}} \right),$$

is the concatenation of $2^{2^q \left(\frac{q-1}{2} \right)_k}$ simplex codes of type β over R_1 , denoted by $S_{(1,k)}^\beta$.

Proof. The proof is by induction on q and Theorem 4.6. For $q = 2$, $G_{(2,k)}^\beta$ is a generator matrix for the simplex codes of type β over R_2 . Then

$$\Gamma_2 \left(G_{(2,k)}^\beta \right) = \left(\overbrace{G_{(1,k)}^\beta \mid G_{(1,k)}^\beta \mid \cdots \mid G_{(1,k)}^\beta}^{2^{2k}} \right),$$

where $G_{(1,k)}^\beta$ is a generator matrix for the simplex code of type β over R_1 . Then

$$\Gamma_{q-1} \left(\Gamma_{q-2} \cdots \left(\Gamma_2 \left(G_{(2,k)}^\beta \right) \right) \right) = \left(2^{2^{q-1}k} \cdots \left(2^{2^2k} G_{(1,k)}^\beta \right) \right),$$

is the generator matrix obtained by the concatenation of $2^{2^q \left(\frac{q-1}{2} \right)_k}$ $S_{(1,k)}^\beta$ codes, where $S_{(1,k)}^\beta$ is the simplex code of type β over R_1 . Then

$$\Gamma_q \left(\Gamma_{q-1} \cdots \left(\Gamma_2 \left(S_{(2,k)}^\beta \right) \right) \right) = \left(\overbrace{S_{(1,k)}^\beta \mid S_{(1,k)}^\beta \mid \cdots \mid S_{(1,k)}^\beta}^{2^{2^{(q-2) \left(\frac{q+1}{2} \right)_k}}} \right).$$

\square

4.1 Binary Gray Images of the Simplex Codes of Type β

The binary images of the simplex codes of type β over R_q are given in the following theorems.

Theorem 4.8 *Let $S_{(q,k)}^\beta$ be the simplex code over R_q of length $2^{(2^{2^q}-1)(k-1)}(2^k-1)$, 2-dimension $2^q k$ and minimum Lee weight d_{Lee} . Then $\Psi_{Lee}(S_{(q,k)}^\beta)$ is the concatenation of $2^{(2^{2^q}-1)(k-1)+q}$ simplex codes with parameters $[2^{(2^{2^q}-1)(k-1)+q}(2^k-1); k; d_{Ham} = 2^{(2^{2^q}-2)k+q}]$.*

Proof. If $G_{(q,k)}^\beta$ is a generator matrix of the simplex code $S_{(q,k)}^\beta$ over R_q , then $\Psi_{Lee}(G_{(q,k)}^\beta)$ has the following form

$$\Psi_{Lee}(G_{(q,k)}^\beta) = \left(\overbrace{G_k \mid G_k \mid \cdots \mid G_k}^{2^{(2^{2^q}-1)(k-1)+q}} \right),$$

where G_k is a generator matrix of the binary simplex code S_k . The result then follows by induction on k . \square

Theorem 4.9 *Let $S_{(q,k)}^\beta$ be the simplex code over R_q of length $2^{(2^{2^q}-1)(k-1)}(2^k-1)$, 2-dimension $2^q k$ and minimum homogeneous weight d_{hom} . Then $\Psi_{hom}(S_{(q,k)}^\beta)$ is the concatenation of $2^{(2^{2^q}-1)(k-1)+(q+1)}$ binary simplex codes with parameters $[2^{(2^{2^q}-1)(k-1)+(q+1)}(2^k-1); k; d_{Ham} = 2^{(2^{2^q}-2)(k-1)+(q+1)}]$.*

Proof. The proof is similar to that of Theorem 4.8. \square

5 MacDonald Codes of Types α and β over R_q

In [12], the MacDonald code $\mathcal{M}_{k,u}(q)$ over the finite field \mathbb{F}_q was defined as the unique $\left[\frac{q^k - q^u}{q-1}, k, q^{k-1} - q^{u-1} \right]$ code in which every nonzero codeword has weight either q^{k-1} or $q^{k-1} - q^{u-1}$.

Let $G_{(q,k)}^\alpha$ and $G_{(q,k)}^\beta$ be the generator matrices of the simplex codes of types α and β over R_q , respectively. For $1 \leq u \leq k-1$, we define $G_{(q,k,u)}^\alpha$ (resp. $G_{(q,k,u)}^\beta$), as the generator matrix of the MacDonald code $\mathcal{M}_{(q,k,u)}^\alpha$ (resp. $\mathcal{M}_{(q,k,u)}^\beta$), obtained from $G_{(q,k)}^\alpha$ (resp. $G_{(q,k)}^\beta$), by deleting the columns corresponding to the columns of $G_{(q,u)}^\alpha$ and $0_{2^{2^q}u \times (k-u)}$ (resp. $G_{(q,u)}^\beta$ and $0_{2^{(2^q-1)(u-1)}(2^u-1) \times (k-u)}$), given by

$$G_{(q,k,u)}^\alpha = \left(G_{(q,k)}^\alpha \setminus \frac{0_{2^{2^q}u \times (k-u)}}{G_{(q,u)}^\alpha} \right), \quad (8)$$

and

$$G_{(q,k,u)}^\beta = \left(G_{(q,k)}^\beta \quad \setminus \quad \frac{0_{2^{(2^q-1)(u-1)}(2^u-1) \times (k-u)}}{G_{(q,u)}^\beta} \right). \quad (9)$$

The code $\mathcal{M}_{(q,k,u)}^\alpha$ (resp. $\mathcal{M}_{(q,k,u)}^\beta$), generated by $G_{(q,k,u)}^\alpha$ (resp. $G_{(q,k,u)}^\beta$), is a punctured code of $S_{(q,k)}^\alpha$ (resp. $S_{(q,k)}^\beta$), and is the MacDonald code of type α (resp. β). The MacDonald code $\mathcal{M}_{(q,k,u)}^\alpha$ is a code over R_q of length $2^{2^q k} - 2^{2^q u}$ and 2-dimension $2^q k$. The MacDonald code $\mathcal{M}_{(q,k,u)}^\beta$ is a code over R_q of length $2^{(2^q-1)(k-1)}(2^k - 1) - 2^{(2^q-1)(u-1)}(2^u - 1)$ and 2-dimension $2^q k$.

For example, if $q = 2$, $k = 3$ and $1 \leq u \leq 2$, there are two MacDonald codes of type α ($\mathcal{M}_{(2,3,1)}^\alpha$ and $\mathcal{M}_{(2,3,2)}^\alpha$), and two MacDonald codes of type β ($\mathcal{M}_{(2,3,1)}^\beta$ and $\mathcal{M}_{(2,3,2)}^\beta$). If $\mathcal{U}_{\{1,2\}} = 1 + u_1 + u_2 + u_1 u_2$ and $\mathcal{V}_{\{1,2\}} = u_1 + u_2 + u_1 u_2$, then the generator matrices of these codes are given by

$$G_{(2,3,1)}^\alpha = \left(\begin{array}{c|c|c|c} \overbrace{1 \cdots 1}^{256} & \overbrace{u_1 \cdots u_1}^{256} & \cdots & \overbrace{\mathcal{U}_{\{1,2\}} \cdots \mathcal{U}_{\{1,2\}}}^{256} \\ \hline G_{(2,2)}^\alpha & G_{(2,2)}^\alpha & \cdots & G_{(2,2)}^\alpha \end{array} \right),$$

$$G_{(2,3,2)}^\alpha = \left(\begin{array}{c|c|c|c} \overbrace{0 \cdots 0}^{240} & \overbrace{1 \cdots 1}^{256} & \cdots & \overbrace{\mathcal{U}_{\{1,2\}} \cdots \mathcal{U}_{\{1,2\}}}^{256} \\ \hline \overbrace{1 \cdots 1}^{16} \cdots \overbrace{\mathcal{U}_{\{1,2\}} \cdots \mathcal{U}_{\{1,2\}}}^{16} & \overbrace{0 \cdots 0}^{16} \cdots \overbrace{\mathcal{U}_{\{1,2\}} \cdots \mathcal{U}_{\{1,2\}}}^{16} & \cdots & \overbrace{0 \cdots 0}^{16} \cdots \overbrace{\mathcal{U}_{\{1,2\}} \cdots \mathcal{U}_{\{1,2\}}}^{16} \\ \hline G_{(2,1)}^\alpha \setminus \overbrace{01 \cdots \mathcal{U}_{\{1,2\}}}^{16} & G_{(2,1)}^\alpha & \cdots & G_{(2,1)}^\alpha \end{array} \right)$$

$$G_{(2,3,1)}^\beta = \left(\begin{array}{c|c|c|c} \overbrace{1 \cdots 1}^{256} & \overbrace{0 \cdots 0}^{23} & \cdots & \overbrace{\mathcal{V}_{\{1,2\}} \cdots \mathcal{V}_{\{1,2\}}}^{24} \\ \hline \overbrace{0 \cdots 0}^{16} \cdots \overbrace{\mathcal{U}_{\{1,2\}} \cdots \mathcal{U}_{\{1,2\}}}^{16} & \overbrace{1 \cdots 1}^{16} \overbrace{u_1 \cdots \mathcal{V}_{\{1,2\}}}^7 & \cdots & \overbrace{1 \cdots 1}^{16} \overbrace{0 u_1 \cdots \mathcal{V}_{\{1,2\}}}^8 \\ \hline \overbrace{G_{(2,1)}^\alpha \cdots G_{(2,1)}^\alpha}^{16} & \overbrace{G_{(2,1)}^\alpha}^{16} \overbrace{1 \cdots 1}^7 & \cdots & \overbrace{G_{(2,1)}^\alpha}^{16} \overbrace{1 \cdots 1}^8 \end{array} \right)$$

$$G_{(2,3,2)}^\beta = \left(\begin{array}{c|c|c|c} \overbrace{1 \cdots 1}^{256} & \overbrace{u_1 \cdots u_1}^{24} & \cdots & \overbrace{\mathcal{V}_{\{1,2\}} \cdots \mathcal{V}_{\{1,2\}}}^{24} \\ \hline G_{(2,2)}^\alpha & G_{(2,2)}^\beta & \cdots & G_{(2,2)}^\beta \end{array} \right)$$

Using the previous notation, we have the following results.

Theorem 5.1 *Let $\mathcal{M}_{(q,k,u)}^\alpha$ and $\mathcal{M}_{(q,k,u)}^\beta$ be the MacDonald codes of types α and β , respectively, over R_q . Then $\Gamma_q(\mathcal{M}_{(q,k,u)}^\alpha)$ and $\Gamma_q(\mathcal{M}_{(q,k,u)}^\beta)$ are the concatenation of $2^{2^q-1}k$ MacDonald codes of types α and β , respectively, over R_{q-1} .*

Proof. The proof is similar to those for Theorems 3.5 and 4.6. \square

Theorem 5.2 *If $\mathcal{M}_{(q,k,u)}^\alpha$ is the MacDonald code of type α over R_q , then*

$$\Gamma_q \left(\Gamma_{q-1} \cdots \left(\Gamma_2 \left(\mathcal{M}_{(2,k,u)}^\alpha \right) \right) \right) = \left(\mathcal{M}_{(1,k,u)}^\alpha \mathcal{M}_{(1,k,u)}^\alpha \cdots \mathcal{M}_{(1,k,u)}^\alpha \right)$$

is the concatenation of $2^{2^q \left(\frac{q-1}{2} \right)^k}$ $\mathcal{M}_{(1,k,u)}^\alpha$ codes, where $\mathcal{M}_{(1,k,u)}^\alpha$ is the MacDonald code of type α over R_1).

If $\mathcal{M}_{(q,k,u)}^\beta$ is the MacDonald code of type β over R_q , then

$$\left(\Gamma_q \left(\Gamma_{q-1} \cdots \left(\Gamma_2 \left(\mathcal{M}_{(2,k,u)}^\beta \right) \right) \right) \right) = \left(\mathcal{M}_{(1,k,u)}^\beta \mathcal{M}_{(1,k,u)}^\beta \cdots \mathcal{M}_{(1,k,u)}^\beta \right),$$

is the concatenation of $2^{2^q \left(\frac{q-1}{2} \right)^k}$ copies of $\mathcal{M}_{(1,k,u)}^\beta$, where $\mathcal{M}_{(1,k,u)}^\beta$ is the MacDonald code of type β over R_1 .

Proof. The proof is similar to those for Theorems 3.6 and 4.7. □

In the remainder of this paper, we denote by $\mathcal{M}_{T,\alpha}$ and $\mathcal{M}_{T,\beta}$ the torsion codes of $\mathcal{M}_{(q,k,u)}^\alpha$ and $\mathcal{M}_{(q,k,u)}^\beta$, respectively. Next, the Hamming weight distributions of $\mathcal{M}_{T,\alpha}$ and $\mathcal{M}_{T,\beta}$ are obtained.

Theorem 5.3 *The torsion code $\mathcal{M}_{T,\alpha}$ is a linear code with parameters $(2^{2^q k} - 2^{2^q u}, k; 2^{2^q k-1} - 2^{2^q u-1})$. The number of codewords with Hamming weight $2^{2^q k-1} - 2^{2^q u-1}$ is equal to $2^k - 2^{k-u}$, the number of codewords with Hamming weight $2^{2^q k-1}$ is equal to $2^{k-u} - 1$, and there is one codeword of zero weight.*

Proof. The generator matrix of the torsion code $\mathcal{M}_{T,\alpha}$ is obtained by replacing $u_1 u_2 \cdots u_q$ by 1 in the matrix $u_1 u_2 \cdots u_q G_{(q,k,u)}^\alpha$. Similar to the proof of [1, Lemma 3.1], the proof is by induction on k and u . It is clear that the result holds for $k = 2$ and $u = 1$. Suppose the result holds for $k - 1$ and $1 \leq u \leq k - 2$. Then for k and $1 \leq u \leq k - 1$, the matrix $u_1 u_2 \cdots u_q G_{(q,k,u)}^\alpha$ has the form

$$u_1 u_2 \cdots u_q G_{(q,k,u)}^\alpha = \left(u_1 u_2 \cdots u_q G_{(q,k)}^\alpha \quad \setminus \quad \frac{0_{2^{2^u} \times (k-u)}}{u_1 u_2 \cdots u_q G_{(q,u)}^\alpha} \right). \quad (10)$$

Then each nonzero codeword of $u_1 u_2 \cdots u_q G_{(q,k,u)}^\alpha$ has Hamming weight $2^{2^q k-1} - 2^{2^q u-1}$ or $2^{2^q k-1}$, and the dimension of the torsion code $\mathcal{M}_{T,\alpha}$ is k . Hence, the number of codewords with Hamming weight $2^{2^q k-1} - 2^{2^q u-1}$ is $2^k - 2^{k-u}$, and the number of codewords with Hamming weight $2^{2^q k-1}$ is $2^{k-u} - 1$. □

Theorem 5.4 *The Hamming, Lee and homogeneous weight distributions of $\mathcal{M}_{(q,k,u)}^\alpha$ are*

$$(i) \ A_{Ham}(0) = 1, \ A_{Ham}(2^{2^q k-1} - 2^{2^q u-1}) = 2^k - 2^{k-u}, \text{ and } A_{Ham}(2^{2^q k-1}) = 2^{k-u} - 1.$$

- (ii) $A_{Lee}(0) = 1$, $A_{Lee}(2^{2^q k+1}) = 2^{2^q(k-u)} - 1$, and $A_{Lee}(2^{2^q k+1} - 2^{2^q u+1}) = 2^{2^q(k-u)}(2^{2^q u} - 1)$.
- (iii) $A_{hom}(0) = 1$, $A_{hom}(2^{2^q k+1}) = 2^{2^q(k-u)} - 1$, and $A_{hom}(2^{2^q k+1} - 2^{2^q u+1}) = 2^{2^q(k-u)}(2^{2^q u} - 1)$.

Proof. By Lemma 3.3 and (8), there are codewords of $\mathcal{M}_{(q,k,u)}^\alpha$ with Hamming weight $2^{2^q k-1} - 2^{2^q u-1}$ or $2^{2^q k-1}$, and Lee and homogeneous weights $2^{2^q k+1}$ or $2^{2^q k+1} - 2^{2^q u+1}$. Furthermore, by Theorem 5.3 the dimension of the torsion code $\mathcal{M}_{T,\alpha}$ is k . Thus we have $2^{k-u} - 1$ codewords of Hamming weight $2^{2^q k-1}$ and $2^{2^q k-1} - 2^{2^q u-1}$ codewords of Hamming weight $2^k - 2^{k-u}$. \square

Theorem 5.5 *The torsion code $\mathcal{M}_{T,\beta}$ is a linear code with parameters $(2^{(2^q-1)(k-1)}(2^k-1) - 2^{(2^q-1)(u-1)}(2^u-1); k; 2^{2^q k-2^q} - 2^{2^q u-2^q})$. The number of codewords with Hamming weight $2^{2^q k-2^q} - 2^{2^q u-2^q}$ is $2^k - 2^{k-u}$, the number of codewords with Hamming weight $2^{2^q k-2^q}$ is $2^{k-u} - 1$, and there is one codeword of weight 0.*

Proof. The proof is similar to that for Theorem 5.3. \square

5.1 Binary Gray Images of MacDonald Codes of Types α and β over R_q

The binary Gray images of the MacDonald codes of types α and β are considered in this section.

5.1.1 Binary Gray Images of MacDonald Codes of Type α

We now determine the binary images of the MacDonald codes of type α over R_q . The first theorem considers the Lee weight and the second theorem considers the homogeneous weight.

Theorem 5.6 *Let $\mathcal{M}_{(q,k,u)}^\alpha$ be the MacDonald code of type α over R_q of length $2^{2^q k} - 2^{2^q u}$, 2-dimension $2^q k$ and minimum Lee weight d_{Lee} . Then $\Psi_{Lee}(S_{(q,k)}^\alpha)$ is the concatenation of $\frac{2^{2^q k+q} - 2^{2^q u+q}}{2^k - 2^u}$ binary MacDonald codes with parameters $[2^{2^q k+q} - 2^{2^q u+q}; k; d_{Ham} = 2^{2^q k+q-1} - 2^{2^q u+q-1}]$.*

Proof. The proof is similar to that of Theorem 3.9. \square

Theorem 5.7 *Let $\mathcal{M}_{(q,k,u)}^\alpha$ be the MacDonald code of type α over R_q of length $2^{2^q k} - 2^{2^q u}$, 2-dimension $2^q k$ and minimum homogeneous weight d_{hom} . Then $\Psi_{hom}(S_{(q,k)}^\alpha)$ is the*

concatenation of $\frac{2^{2^q k + q + 1} - 2^{2^q u + q + 1}}{2^k - 2^u}$ binary MacDonald codes with parameters $[2^{2^q k + q + 1} - 2^{2^q u + q + 1}; k; d_{Ham} = 2^{2^q k + q} - 2^{2^q u + q}]$.

Proof. The proof is similar to that of Theorem 3.9. \square

5.1.2 Binary Gray Images of MacDonald Codes of Types β

The binary Gray images of the MacDonald codes of type β are now given.

Theorem 5.8 Let $\mathcal{M}_{(q,k,u)}^\beta$ be the MacDonald code of type β over R_q of length $2^{(2^q-1)(k-1)}(2^k-1) - 2^{(2^q-1)(u-1)}(2^u-1)$, 2-dimension $2^q k$ and minimum Lee weight d_{Lee} . Then $\Psi_{Lee}(S_{(q,k)}^\beta)$ is the concatenation of $\frac{2^{(2^q-1)(k-1)+q}(2^k-1) - 2^{(2^q-1)(u-1)+q}(2^u-1)}{2^k - 2^u}$ copies of the binary MacDonald code with parameters $[2^{(2^q-1)(k-1)+q}(2^k-1) - 2^{(2^q-1)(u-1)+q}(2^u-1); k; d_{Ham} = 2^{(2^q-1)(k-1)+q-1}(2^k-1) - 2^{(2^q-1)(u-1)+q-1}(2^u-1)]$.

Proof. The proof is similar to that of Theorem 3.9. \square

Theorem 5.9 Let $\mathcal{M}_{(q,k,u)}^\beta$ be the MacDonald code of type β over R_q of length $2^{(2^q-1)(k-1)}(2^k-1) - 2^{(2^q-1)(u-1)}(2^u-1)$, 2-dimension $2^q k$ and minimum homogeneous weight d_{hom} . Then $\Psi_{hom}(S_{(q,k)}^\beta)$ is the concatenation of $\frac{2^{(2^q-1)(k-1)+(q+1)}(2^k-1) - 2^{(2^q-1)(u-1)+(q+1)}(2^u-1)}{2^k - 2^u}$ binary MacDonald codes with parameters $[2^{(2^q-1)(k-1)+(q+1)}(2^k-1) - 2^{(2^q-1)(u-1)+(q+1)}(2^u-1); k; d_{Ham} = 2^{(2^q-1)(k-1)+q}(2^k-1) - 2^{(2^q-1)(u-1)+q}(2^u-1)]$.

Proof. The proofs are similar to those for Theorems 3.10 and 4.9. \square

6 The Repetition Codes over R_q and their Covering Radius

The repetition code C over a finite field \mathbb{F}_q is an $[n; 1; n]$ linear code. The covering radius of C is $\lfloor \frac{n(q-1)}{q} \rfloor$ [7]. We begin by defining the repetition codes over R_q . Let

$$\mathcal{U}_A = \left(1 + \sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset = 0 \vee A \neq \emptyset}} c_A u_A \right),$$

and

$$\mathcal{V}_A = \sum_{\substack{A \subseteq \{1,2,\dots,q\} \\ c_\emptyset = 0 \vee A \neq \emptyset}} c_A u_A.$$

Two types of repetition codes can be defined over R_q .

Type 1 The repetition codes C_c generated by

$$G_c = \left(\overbrace{cc \cdots c}^n \right),$$

where c is an element of $R_q - \{0, u_1 u_2 \cdots u_q\}$.

Type 2 The repetition codes $C_{u_1 u_2 \cdots u_q}$ generated by

$$G_{u_1 u_2 \cdots u_q} = \left(\overbrace{u_1 u_2 \cdots u_q u_1 u_2 \cdots u_q \cdots u_1 u_2 \cdots u_q}^n \right).$$

Theorem 6.1 *The covering radius of the repetition codes over R_q is given by*

$$(i) \ r_{hom}(C_c) = 2^q n \text{ and } r_{Lee}(C_c) = 2^q n.$$

$$(ii) \ r_{hom}(C_{u_1 u_2 \cdots u_q}) = 2^{q+1} n \text{ and } r_{Lee}(C_{u_1 u_2 \cdots u_q}) = 2^q n.$$

Proof. For part (i), by definition $r_{hom}(C_c) = \max_{x \in (R_q)^n} d\{x, C_c\}$. Let $x \in (R_q - \{0, u_1 u_2 \cdots u_q\})^n$. Then as a direct consequence, for all $y \in C_c$ we have $d\{x, y\} = 2^q n$, so that $r_{hom}(C_c) = 2^q n$. By Proposition 2.10, we obtain that $r_{Lee}(C_c) = r_{Ham}(\Psi_{Lee}(C_c)) = 2^q n$. The proof of part (ii) is similar. \square

Let C be the linear code over R_q generated by the matrix

$$G = \left(\overbrace{11 \cdots 1}^n \overbrace{u_1 u_1 \cdots u_1}^n \cdots \overbrace{u_A u_A \cdots u_A}^n \right).$$

Then C is the repetition code of length $(2^{2^q} - 1)n$.

Theorem 6.2 *A linear code C generated by the matrix*

$$G = \left(\overbrace{11 \cdots 1}^n \overbrace{u_1 u_1 \cdots u_1}^n \cdots \overbrace{u_A u_A \cdots u_A}^n \right),$$

has covering radius given by

$$r_{hom}(C) = 2^{2^q+q} n \text{ and } r_{Lee}(C) = (2^{2^q} - 1) 2^{q-1} n.$$

Proof. The vectors of C generated by G can be divided into three classes.

(1) The vectors of C with components from all the element of R_q

$$x_a = (x_1 x_2 \cdots x_n) \in C, x_i \in R_q \text{ for all } 1 \leq i \leq n.$$

(2) The vectors of C with components that are zero divisors of R_q

$$x_b = (x_1 x_2 \cdots x_n) \in C, x_i \in \mathfrak{D}(R_q) \text{ for all } 1 \leq i \leq n.$$

(3) The vectors of C with components 0 or $u_1 u_2 \cdots u_q$

$$x_c = (x_1 x_2 \cdots x_n) \in C, x_i \in \{0, u_1 u_2 \cdots u_q\} \text{ for all } 1 \leq i \leq n.$$

For $x \in (R_q)^n$, we have that $d(x, x_a) = d(x, x_b) = d(x, x_c) = 2^{2^q+q}n$, so $r_{hom}(C) \geq 2^{2^q+q}n$. On the other hand, for class (1), if $x = (11 \cdots 1) \in (R_q)^n$ and $x_a = (1u_1 \cdots \mathcal{U}_A) \in (R_q)^n$, then $x + x_a = (0(1 + u_1) \cdots \mathcal{V}_A)$ is a permutation equivalent to x_a so that

$$x + x_a = \sigma(x_a).$$

Then $d(x, x_a) \leq 2^{2^q+q}n$, and hence $r_{hom}(C) \leq 2^{2^q+q}n$. For class (2), if $x = (11 \cdots 1) \in (R_q)^n$ and $x_b = (u_1 u_2 \cdots \mathcal{V}_A) \in (\mathfrak{D}(R_q))^n$, so that

$$x + x_b = ((1 + u_1)(1 + u_2) \cdots \mathcal{U}_A) \in (\mathfrak{U}(R_q))^n.$$

Then $d(x, x_b) \leq 2^{2^q+q}n$, and hence $r_{hom}(C) \leq 2^{2^q+q}n$. For class (3), if $x = (11 \cdots 1) \in (R_q)^n$ and $x_c = (0(u_1 u_2 \cdots u_q) \cdots (u_1 u_2 \cdots u_q)) \in (\mathfrak{D}(R_q))^n$, then

$$x + x_c = (1(1 + u_1 u_2 \cdots u_q) \cdots (1 + u_1 u_2 \cdots u_q)) \in (\mathfrak{U}(R_q))^n,$$

so that $d(x, x_c) \leq 2^{2^q+q}n$ and hence $r_{hom}(C) \leq 2^{2^q+q}n$.

By Proposition 2.10 we then have that $r_{Lee}(C) = r_{Ham}(\Psi_{Lee}(C)) = (2^{2^q} - 1)2^{q-1}n$. \square

7 The Covering Radius of Simplex and MacDonald Codes of Types α and β over R_q

We now determine the covering radius of simplex and MacDonald codes of types α and β over R_q . This requires the covering radius of the repetition code over R_q .

7.1 The Covering Radius of Simplex Codes of Types α and β over R_q

The covering radius of simplex codes of types α and β over R_q is given by the following theorems.

Theorem 7.1 *The covering radius of the simplex codes of type α over R_q with respect to the homogeneous and Lee weights is*

$$(i) \ r_{hom}(S_{(q,k)}^\alpha) = k \cdot 2^{2^q k + q}.$$

$$(ii) \ r_{Lee}(S_{(q,k)}^\alpha) = 2^{(2^q+1)k+1}.$$

Proof. For part (i), if $x \in (R_q)^n$, we have $d_{hom}(x, S_{(q,k)}^\alpha) = k \cdot 2^{2^q k + q}$. Hence by definition, $r_{hom}(S_{(q,k)}^\alpha) \geq k \cdot 2^{2^q k + q}$. On the other hand, applying Proposition 2.11 and Theorem 6.2 gives

$$\begin{aligned} r_{hom}(S_{(q,k)}^\alpha) &\leq r_{hom} \left(\left[\overbrace{11 \cdots 1}^{2^{2^q}(k-1)} \overbrace{u_1 u_1 \cdots u_1}^{2^{2^q}(k-1)} \cdots \overbrace{\mathcal{U}_A \mathcal{U}_A \cdots \mathcal{U}_A}^{2^{2^q}(k-1)} \right] \right) + 2^{2^q} \cdot r_{hom}(S_{(q,k-1)}^\alpha) \\ &\leq 2^{2^q k + q} + 2^{2^q(k-1)+q} \cdot 2^{2^q} + \cdots + 2^{q \cdot 2^q} \cdot r_{hom}(S_{(q,1)}^\alpha) \\ &\leq 2^{2^q k + q} + 2^{2^q(k-1)+q} \cdot 2^{2^q} + \cdots + 2^{2^q(k-q)+q} \cdot 2^{q \cdot 2^q} \\ &\leq k \cdot 2^{2^q k + q}. \end{aligned}$$

For part (ii), from Proposition 2.10 we have

$$r_{Lee}(S_{(q,k)}^\alpha) = r_{Ham}(\Psi_{Lee}(S_{(q,k)}^\alpha)) = 2^{(2^q+1)k+1}.$$

□

Theorem 7.2 *The covering radius of the simplex codes of type β over R_q with respect the homogeneous and Lee weights is*

$$(i) \ r_{hom}(S_{(q,k)}^\beta) = 2^{2^q(k-2)+q} [2^{2^q}(k - 2^{-q}) + 4 - 2^{-q+1}].$$

$$(ii) \ r_{Lee}(S_{(q,k)}^\beta) = 2^{(2^q-1)(k-1)+(q-1)}(2^k - 1).$$

Proof. For part (i), if $x \in (R_q)^n$, we have $d_{hom}(x, S_{(q,k)}^\beta) = 2^{2^q(k-2)+q} [2^{2^q}(k - 2^{-q}) + 4 - 2^{-q+1}]$. Hence by definition, $r_{hom}(S_{(q,k)}^\beta) \geq 2^{2^q(k-2)+q} [2^{2^q}(k - 2^{-q}) + 4 - 2^{-q+1}]$. On the other hand, applying Proposition 2.11 and Theorem 6.2 gives

$$\begin{aligned} r_{hom}(S_{(q,k)}^\beta) &\leq r_{hom} \left(\left[\overbrace{1 \cdots 1}^{2^{2^q}(k-1)} \cdots \overbrace{\mathcal{V}_A \cdots \mathcal{V}_A}^{2^{(2^q-1)(k-1)}(2^k-1)} \right] \right) + r_{hom}(S_{(q,k-1)}^\alpha) + 2^{2^q-1} \cdot r_{hom}(S_{(q,k-1)}^\beta) \\ &\leq 2^{2^q(k-2)+q} (2^{2^q-1} + 2) + \cdots + 2^{2^q(k-2)+q}(k-1) + 2^{q \cdot 2^q - q} \cdot r_{hom}(S_{(q,2)}^\beta) \\ &\leq 2^{2^q(k-2)+q} (2^{2^q-1} + 2)(2 - 2^{-q}) + \cdots + 2^{2^q(k-2)+q}(k-1) \\ &\leq 2^{2^q(k-2)+q} [2^{2^q}(k - 2^{-q}) + 4 - 2^{-q+1}]. \end{aligned}$$

Then similar to the proof of part (ii) of Theorem 7.1, the result follows. \square

7.2 Covering Radius of MacDonald Codes of Types α and β over R_q

The covering radius of the MacDonald codes of types α and β over R_q is given by the following theorems.

Theorem 7.3 *The covering radius of the MacDonald codes of type α over R_q with respect to the homogeneous and Lee weights is*

$$(i) \text{ For } u \leq e \leq k, r_{hom}(\mathcal{M}_{(q,k,u)}^\alpha) \leq 2^{2^q k} - 2^{2^q u} + r_{hom}(\mathcal{M}_{(q,e,u)}^\alpha).$$

$$(ii) \text{ } r_{Lee}(\mathcal{M}_{(q,k,u)}^\alpha) = 2^{2^q k + (q-1)} - 2^{2^q u + (q-1)}.$$

Proof. For the first part, from Proposition 2.11 and Theorem 6.2, if $u \leq e \leq k$, we have

$$\begin{aligned} r_{hom}(\mathcal{M}_{(q,k,u)}^\alpha) &\leq (2^{2^q} - 1)(2^{2^q k - 2^q}) + r_{hom}(\mathcal{M}_{(q,k-1,u)}^\alpha) \\ &\leq (2^{2^q} - 1)(2^{2^q k - 2^q}) + (2^{2^q} - 1)(2^{2^q k - (2^q - 2)}) + \dots + (2^{2^q} - 1)2^{2^q e} \\ &\quad + r_{hom}(\mathcal{M}_{(q,e,u)}^\alpha) \\ &\leq 2^{2^q k} - 2^{2^q e} + r_{hom}(\mathcal{M}_{(q,e,u)}^\alpha). \end{aligned}$$

For the second part, by Proposition 2.10, we obtain that

$$r_{Lee}(\mathcal{M}_{(q,k,u)}^\alpha) = r_{Ham}(\Psi_{Lee}(\mathcal{M}_{(q,k,u)}^\alpha)) = 2^{2^q k + (q-1)} - 2^{2^q u + (q-1)}.$$

\square

Theorem 7.4 *The covering radius of the MacDonald codes of type β over R_q with respect to the homogeneous and Lee weights is*

$$(i) \text{ For } u \leq e \leq k, r_{hom}(\mathcal{M}_{(q,k,u)}^\beta) \leq 2^{(2^q-1)(k-1)}(2^k-1) - 2^{(2^q-1)(u-1)}(2^u-1) + r_{hom}(\mathcal{M}_{(q,e,u)}^\beta).$$

$$(ii) \text{ } r_{Lee}(\mathcal{M}_{(q,k,u)}^\beta) = 2^{(2^q-1)(k-1)+(q-1)}(2^k-1) - 2^{(2^q-1)(u-1)+(q-1)}(2^u-1).$$

Proof. For the first part, from Proposition 2.11 and Theorem 6.2, if $u \leq e \leq k$, we have

$$\begin{aligned} r_{hom}(\mathcal{M}_{(q,k,u)}^\beta) &\leq (2^{2^q} - 1)2^{(2^q-1)(k-1)-(2^q-1)}(2^k-1) + r_{hom}(\mathcal{M}_{(q,k-1,u)}^\beta) \\ &\leq (2^{2^q} - 1)2^{(2^q-1)(k-1)-(2^q-1)}(2^k-1) + (2^{2^q} - 1)2^{(2^q-1)(k-1)-((2^q-1)-2)}(2^k-1) \\ &\quad + \dots + (2^{2^q} - 1)2^{(2^q-1)(k-1)-(2^e-1)}(2^k-1) + r_{hom}(\mathcal{M}_{(q,e,u)}^\beta) \\ &\leq 2^{(2^q-1)(k-1)}(2^k-1) - 2^{(2^q-1)(e-1)}(2^e-1) + r_{hom}(\mathcal{M}_{(q,e,u)}^\beta). \end{aligned}$$

For the second part, By Proposition 2.10, we obtain that

$$r_{Lee}(\mathcal{M}_{(q,k,u)}^\beta) = r_{Ham}(\Psi_{Lee}(\mathcal{M}_{(q,k,u)}^\beta)) = 2^{(2^q-1)(k-1)+(q-1)}(2^k - 1) - 2^{(2^q-1)(u-1)+(q-1)}(2^u - 1).$$

□

References

- [1] M. Al-Ashker, *Simplex codes over the ring $\sum_{n=0}^s u^n \mathbb{F}_2$* , Turk. J. Math., vol. 29, pp. 221–233, 2005.
- [2] T. Aoki, P. Gaborit, M. Harada, M. Ozeki, and P. Solé, *On the covering radius of \mathbb{Z}_4 -codes and their lattices*, IEEE Trans. Inform. Theory, vol. 45, no. 6, pp. 2162–2168, 1999.
- [3] G.D. Cohen, M.G. Karpovsky, H.F. Mattson, and J.R. Schatz, *Covering radius - Survey and recent results*, IEEE Trans. Inform. Theory, vol. 31, no. 3, pp. 328–343, 1985.
- [4] S.T. Dougherty, T.A. Gulliver, and J. Wong, *Self-dual codes over \mathbb{Z}_8 and \mathbb{Z}_9* , Designs, Codes, Crypt., 41, pp. 235–249, 2006.
- [5] S.T. Dougherty, B. Yildiz, and S. Karadeniz, *Codes over R_k , Gray maps and their binary images*, Finite Fields Appl., vol. 17, no. 3, pp. 205–219, May 2011.
- [6] M. Greferath and S. E. Schmidt, *Finite-ring combinatorics and MacWilliams equivalence theorem*, J. Combin. Theory Ser. A, 92, pp. 17–28, 2000.
- [7] M.K. Gupta and C. Durairajan, *On the covering radius of Some modular codes*, Adv. Math. Commun., vol. 8, no. 2, pp. 129–137, 2014.
- [8] M.K. Gupta, D.G. Glynn, and T.A. Gulliver, *On senary simplex codes*, Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, Lecture Notes in Computer Science, vol. 2227, pp. 112–121, 2001.
- [9] M.K. Gupta, *On Some Linear Codes over \mathbb{Z}_{2^s}* , Ph.D. Thesis, IIT Kanpur, 1999.
- [10] A.R. Hammons, P.V. Kumar, A.R. Calderbank, N.J.A. Sloane, and P. Solé, *The \mathbb{Z}_4 -linearity of Kerdock, Preparata, Goethals, and related codes*, IEEE Trans. Inform. Theory, vol. 40, pp. 301–319, 1999.
- [11] T. Honold, *Characterization of finite Frobenius rings*, Arch. Math., 76, pp. 406–415, 2001.

- [12] A.M. Patel, *Maximal q -ary codes with large minimum distance*, IEEE Trans. Inform. Theory, 21, pp. 106–110, 1975.
- [13] V.V. Vazirani, H. Sran, and B.S. Rajan, *An efficient algorithm for constructing minimal trellises for codes over finite abelian groups*, IEEE Trans. Inform. Theory, 42(6), pp. 1839–1854, 1996.
- [14] B. Yildiz and S. Karadeniz, *Linear codes over $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$* , Designs, Codes, Crypt., vol. 54, no. 1, pp. 61–81, 2010.
- [15] B. Yildiz and S. Karadeniz, *Cyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$* , Designs, Codes, Crypt., vol. 58, no. 1, pp. 221–234, 2011.
- [16] B. Yildiz and I.G. Kelebek, *The homogeneous weight for R_k , related Gray map and new binary quasicyclic codes*, arXiv:1504.04111v1 [cs.IT] 16 Apr 2015.